SPLITTING OF LOW RANK ACM BUNDLES ON HYPERSURFACES OF HIGH DIMENSION

AMIT TRIPATHI

Abstract. Let X be a smooth projective hypersurface. In this note we show that any rank 3 (resp. rank 4) arithmetically Cohen-Macaulay vector bundle over X splits when $\dim X \geq 7$ (resp. $\dim X \geq 9$).

1. Introduction

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface where $n \geq 3$. By Grothendieck-Lefschetz theorem [7], we know the structure of the set of all line bundles on X. Vector bundles over X are not so well understood. An obvious question about vector bundles on any projective variety is the splitting problem - When can we say that a given vector bundle is a direct sum of line bundles? The proper objects in the category of vector bundles over X to look for the splitting behaviour are arithmetically Cohen-Macaulay bundles. We recall the defintion,

Definition 1.1. *An arithmetically Cohen-Macaulay (ACM) bundle on X is a vector bundle E satisfying*

$$H^i(X, E(m)) = 0, \ \forall\, m \in \mathbb{Z} \text{ and } 0 < i < dim\, X$$

The importance of this definition lies in a well known criterion of Horrocks [10] - ACM bundles are precisely the bundles on $\mathbb{P}^n$ that are split. Viewing $\mathbb{P}^n$ as a hypersurface of degree 1 in $\mathbb{P}^{n+1}$, one may ask if for hypersurfaces with degree $d > 1$, such a splitting holds. When $d > 1$, there exists indecomposable ACM bundles on hypersurfaces (see [13] for a specific example or [15] for a class of examples), though several splitting results are available for various degrees and ranks. In particular, fixing $d = 2$, the ACM bundles on quadrics have been completely classified, see [12]. The case of cubic surfaces in $\mathbb{P}^3$ has been investigated in [3].

In a different direction, we can fix the rank of the bundle and let degree vary. Here the general conjectural picture is that any ACM bundle of a fixed rank, over a sufficiently high dimensional hypersurface (irrespective of its degree) is split. The precise conjecture is,

1991 *Mathematics Subject Classification.* 14J60.

Key words and phrases. Vector bundles, hypersurfaces, arithmetically Cohen-Macaulay.

This work was supported by a postdoctoral fellowship from NBHM, Department of Atomic Energy.

Conjecture (Buchweitz, Greuel and Schreyer [2]*):* Let $X \subset \mathbb{P}^n$ be a hypersurface. Let E be an ACM bundle on X. If rank $E < 2^e$, where $e = \left[\dfrac{n-2}{2}\right]$, then E splits. (Here $[q]$ denotes the largest integer $\leq q$.) □

Splitting of ACM bundles of rank 2 on hypersurfaces have been understood fairly well. We summarize the results known. When $d = 1$, splitting follows by the Horrock's criterion, so we assume $d \geq 2$. Let E be a rank 2 ACM bundle on X, then E splits if,

(1) If $\dim(X) \geq 5$ (see [11] and [13]).
(2) If $\dim(X) = 4$ and X is general hypersurface and $d \geq 3$ (see [13] and [16]).
(3) If $\dim(X) = 3$ and X is general hypersurface and $d \geq 6$ (see [14] and [16]).

The case of a general hypersurface of low degree in $\mathbb{P}^4$ and $\mathbb{P}^5$ have also been studied by Chiantini and Madonna in [4], [5], [6].

For rank ≥ 3, very few results are known. To our knowledge, the only general splitting result in this direction is by Tadakazu [17] who found a splitting criterion for any rank k ACM bundle on a general hypersurface depending on the degree and the dimension of hypersurface.

The conjecture mentioned above predicts that any ACM bundle of rank 3 (resp. rank 4) over a hypersurface in $\mathbb{P}^6$ (resp. $\mathbb{P}^8$) splits. In this note, we prove a weaker version,

Theorem 1.2 (Corollary 3.3 + Corollary 3.4)**.** *Let E be an ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$. Then E splits if,*

(1) *rank $E = 3$ and $n \geq 7$.*
(2) *rank $E = 4$ and $n \geq 9$.*

For rank 2 ACM bundles, our method gives another proof for splitting when $n \geq 5$.

2. Preliminaries

We will work over an algebraically closed field of characteristic zero.

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d \geq 2$. We set a conventional notation

$$H^i_*(X, \mathcal{F}) := \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{F}(m))$$

where $\mathcal{F}$ denotes a coherent sheaf on X.

Let E be a rank k ACM bundle on X. We take a minimal (1-step) resolution of E on $\mathbb{P}^{n+1}$,

$$0 \to \widetilde{F_1} \to \widetilde{F_0} \to E \to 0 \tag{1}$$

where $\widetilde{F_0}$ is direct sum of line bundles on $\mathbb{P}^{n+1}$. By Auslander-Buchsbaum formula, $\widetilde{F_1}$ is a bundle and by Horrock's criterion it is also a split bundle on $\mathbb{P}^{n+1}$.

Restricting (1) to X, we get,

$$0 \to Tor^1_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \to F_1 \to F_0 \to E \to 0 \tag{2}$$

where $F_i = \widetilde{F}_i \otimes \mathcal{O}_X$ for $i = 0, 1$. To compute the Tor term, we tensor the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$ with E,

$$0 \to Tor^1_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \to E(-d) \to E \to E \otimes \mathcal{O}_X \to 0$$

The map $E \to E \otimes \mathcal{O}_X$ is an isomorphism, thus we get $Tor^1_{\mathbb{P}^{n+1}}(E, \mathcal{O}_X) \cong E(-d)$. Exact sequence (2) breaks up into 2 short exact sequences,

$$0 \to G \to F_0 \to E \to 0 \tag{3}$$

$$0 \to E(-d) \to F_1 \to G \to 0 \tag{4}$$

Since $H^0_*(X, F_0) \twoheadrightarrow H^0_*(X, E)$ is a surjection of graded rings, $H^1_*(X, G) = 0$. It follows that G is also ACM.

3. Proof of the main results

Lemma 3.1. *Let E be any non-split bundle (not necessarily ACM) on a hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geq 3$. Assume further that $H^1_*(X, E^\vee) = 0$. Let the exact sequence* (3) *be a minimal (1-step) resolution of E on X, then G does not admit a line bundle as a direct summand.*

Proof. We will assume the contrary. Let $G = G' \oplus L$ where L is a line bundle. By Grothendieck-Lefschetz theorem, L is of the form $\mathcal{O}_X(a)$. There exists following pushout diagram,

$$\begin{tikzcd} 0 \arrow[r] & G' \arrow[r] & F_0' \arrow[r] & E \arrow[r] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[u] & F_0 \arrow[r] \arrow[u] & E \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \tag{5}$$

where $G \to G'$ is the natural projection and F_0' is the pushout. Completion of the diagram (5) gives $\eta : 0 \to L \to F_0 \to F_0' \to 0$. Applying $Hom_X(-, L)$ to the top horizontal sequence gives,

$$\cdots \to Ext^1(E, L) \to Ext^1(F_0', L) \to Ext^1(G', L) \to \cdots$$

In the above sequence $\eta \mapsto \eta'$ where $\eta' : 0 \to L \to G \to G' \to 0$ is split. By assumption $Ext^1(E, L) \cong H^1(X, E^\vee \otimes L) = 0$, thus η splits. Therefore F_0' is a direct sum of line bundles of rank $r - 1$, by Krull-Schmidt theorem [1]. This implies that $0 \to G' \to F_0' \to E \to 0$ is a 1-step resolution of E which contradicts the minimality of the resolution (3). $\square$

On any projective variety Z, for a short exact sequence of vector bundles $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$ and any positive integer k, there exists a resolution of k-th exterior power of $\mathcal{F}_2$,

$$0 \to Sym^k(\mathcal{F}_0) \to \cdots \to Sym^{k-i}(\mathcal{F}_0) \otimes \wedge^i \mathcal{F}_1 \to \cdots \to \wedge^k \mathcal{F}_1 \to \wedge^k \mathcal{F}_2 \to 0$$

We will call this resolution the $Sym - \wedge$ *sequence of index* k [1] associated to the given short exact sequence. There exists a similar resolution of k-th symmetric power of $\mathcal{F}_2$ (by interchanging symmetric product and wedge product) which we will call $\wedge - Sym$ *sequence of index* k associated to the given sequence.

$$0 \to \wedge^k(\mathcal{F}_0) \to \cdots \to \wedge^{k-i}(\mathcal{F}_0) \otimes Sym^i \mathcal{F}_1 \to \cdots Sym^k \mathcal{F}_1 \to Sym^k \mathcal{F}_2 \to 0$$

For details see [18].

We will now prove a result from which Theorem 1.2 will follow.

Theorem 3.2. *Let E be any rank k bundle (not necessarily ACM) on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geq 2k+1$. Assume further that E satisfies the following two conditions,*

1. $H^i_*(X, E) = 0,\ i \in \{2, 3, \ldots, k+1\} \cup \{n-1\}$
2. $H^i_*(X, \wedge^m E) = 0,\ i = 2m-1, 2m, \ldots, k+m$ *for each* $m \in \{2, \ldots, k-1\}$

Then E splits.

Despite the odd assumptions, the proof is very simple and we just use hypothesis of the theorem in $\wedge - Sym$ sequence for various indices to prove certain cohomological vanishings (6), which is then used in a $Sym - \wedge$ sequence to prove the theorem.

Proof of theorem 3.2. We write $\wedge - Sym$ sequence of some index $l \in \{2, \ldots, k\}$ for the short exact sequence (4),

$$\begin{aligned} 0 \to \wedge^l E(-d) \to \wedge^{l-1} E(-d) \otimes F_1 \to \cdots \\ \cdots \to E(-d) \otimes Sym^{l-1} F_1 \to Sym^l F_1 \to Sym^l G \to 0 \end{aligned}$$

This breaks up into short exact sequences,

$$0 \to J_{j-1,l} \to \wedge^{l-j} E(-d) \otimes Sym^j F_1 \to J_{j,l} \to 0$$

where $J_{0,l} = \wedge^l E(-d)$, $J_{j,l}$ is defined inductively for $j = 1, \ldots l-1$ and $J_{l,l} = Sym^l G$. By assumption in the theorem and the fact that F_1 is split, we get $H^i_*(X, J_{j,l}) = 0$, for $i = 2l-j-1, 2l-j, \ldots, k+l-j$. This implies

$$H^i_*(X, Sym^l G) = 0 \text{ for } i = l-1, l, \ldots, k \tag{6}$$

Now we look at $Sym - \wedge$ sequence of the index $k (= \operatorname{rank} E)$ for the sequence (4),

$$0 \to Sym^k G \to Sym^{k-1} G \otimes F_0 \to \cdots G \otimes \wedge^{k-1} F_0 \to \wedge^k F_0 \to \wedge^k E \to 0$$

[1] We were unable to find any standard terminology in the literature for the given resolution.

This breaks up into short exact sequences,

$$0 \to M_{j-1} \to Sym^{k-j}G \otimes \wedge^j F_0 \to M_j \to 0 \tag{7}$$

where $M_0 = Sym^k G$ and M_j is defined inductively for $j = 1, \dots k$ as

$$M_j = \operatorname{coker}(M_{j-1} \to Sym^{k-j}G \otimes \wedge^j F_0)$$

Note that $M_k = \wedge^k E = \mathcal{O}_X(e)$ for some $e \in \mathbb{Z}$. Using the vanishing given by (6) in sequence (7) (and the fact that F_0 are split bundles),

$$H^i_*(X, M_j) = 0 \text{ for } i = k - j - 1, k - j$$

Therefore the short exact sequence $0 \to M_{k-1} \to \wedge^k F_0 \to \wedge^k E \to 0$ splits. In particular, M_{k-1} splits. This implies that the following sequence splits,

$$0 \to M_{k-2} \to G \otimes \wedge^{k-1} F_0 \to M_{k-1} \to 0$$

In particular, G has a line bundle as a direct summand. Thus by lemma 3.1, E splits.

□

Corollary 3.3. *Let E be a rank 3 ACM bundle on a smooth hypersurface X with $dim(X) \geq 7$, then E splits.*

Proof. We note that $\wedge^i E$ is ACM when $i = 1, 2, 3$. In particular, both the assumptions of theorem 3.2 are satisfied. Thus E splits. □

Corollary 3.4. *Let E be a rank 4 ACM bundle on a smooth hypersurface X with $dim(X) \geq 9$, then E splits.*

Proof. As before, we note that $\wedge^i E$ is ACM when $i = 1, 3, 4$. By theorem 3.2, E splits, if we can show that $H^i_*(X, \wedge^2 E) = 0$ for $i = 3, 4, 5, 6$. Since E splits $\Leftrightarrow E^\vee(m)$ splits for some $m \in \mathbb{Z}$, so we can assume that $E^\vee$ is globally generated. Then there exists a section $s \in H^0(X, E^\vee)$ of proper codimension i.e. the zero locus Z of s has codimension 4 in X. This implies that there exists a resolution of $\mathcal{O}_Z$ (see [8], pp. 448),

$$0 \to \wedge^4 E \to \wedge^3 E \to \dots \to E \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

We note that Z is Cohen-Macaulay subscheme of X. A cohomological computation gives $H^i_*(X, \wedge^2 E) = 0$ for $i = 3, 4, 5, 6$ when $\dim(X) \geq 9$. □

Remark: It is easy to verify the hypothesis of theorem 3.2 for any rank 2 ACM bundle when $n \geq 5$ which provides another proof for this well known splitting result.

4. Acknowledgement

I thank G.V Ravindra who introduced me to this area. I thank Jishnu Biswas and Suresh Nayak for providing support and motivation.

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Department of Mathematics, Indian Statistical Institute, Bangalore - 560019, India

E-mail address: `amittr@gmail.com`